The Behavior of High-Dimension Vectors

Recall the Markov's ineq.,for non-negative r.v. $X, a>0$:

$$\mathbb{P}\{X \geqslant a\} \leqslant \frac{\mathbb{E}(X)}{a} \tag{1}$$

Then the Chernoff-method is,for $\lambda > 0$:

$$\begin{aligned}\mathbb{P}\{X \geqslant a\} &= \mathbb{P}\{\exp(\lambda X) \geqslant \exp(\lambda a)\} \\ &= \mathbb{P}\{\exp(\lambda X)\exp(-\lambda a) \geqslant 1\} \\ &\leqslant \mathbb{E}(\exp(\lambda X)\exp(-\lambda a))\end{aligned}$$

Take min at RHS to get a tight upperbound:

$$\mathbb{P}\{X \geqslant a\} \leqslant \min_{\lambda>0} \exp(-\lambda a)\mathbb{E}(\exp(\lambda X)) \tag{2}$$

1 High-dimension behavior

We claim:

Lemme 1. *For* $\{u_i\}^n \overset{i.i.d}{\sim} \mathcal{N}(0, \frac{1}{n}), u = (u_1, \ldots, u_n) \in \mathbb{R}^n, \varepsilon \in (0,1)$*:*

$$\mathbb{P}\{|\|u\|^2 - 1| \geqslant \varepsilon\} \leqslant 2\exp\left(\frac{-n\varepsilon^2}{8}\right), \tag{3}$$

which is also called 'unit-norm lemma'.

Démonstration. Fitst,$\mathbb{P}\{|\|u\|^2 - 1| \geqslant \varepsilon\} = \mathbb{P}\{\|u\|^2 \geqslant \varepsilon + 1\} + \mathbb{P}\{\|u\|^2 \leqslant 1 - \varepsilon\}$,

Now we put our eyes on $\mathbb{P}\{\|u\|^2 \geqslant \varepsilon + 1\}$.By Chernoff-mothod:

$$\mathbb{P}\{\|u\|^2 \geqslant \varepsilon + 1\} \quad \leqslant \min_{\lambda>0} \exp(-\lambda(\varepsilon+1))\mathbb{E}(\exp(\lambda\|u\|^2)) \tag{4}$$

$\mathbb{E}(\exp(\lambda\|u\|^2)$ is the $m.d.f$ of $n\|u\|^2 \sim \chi^2(n) \Rightarrow \mathbb{E}(\exp(\lambda\|u\|^2)) \overset{(*)_1}{=} \left(\frac{n}{n-2\lambda}\right)^{\frac{n}{2}}$.

Then (4) becomes:

$$\begin{aligned}\mathbb{P}\{\|u\|^2 \geqslant \varepsilon + 1\} &\leqslant \min_{\lambda>0} \exp(-\lambda(\varepsilon+1))\left(\frac{n}{n-2\lambda}\right)^{\frac{n}{2}} \\ \text{By derivative, } \lambda = \frac{n\varepsilon}{2(1+\varepsilon)} &\leqslant \exp\left(\frac{n(\log(1+\varepsilon) - \varepsilon)}{2}\right) \\ &\overset{(*)_2}{\leqslant} \exp\left(\frac{-n\varepsilon^2}{8}\right)\end{aligned}$$

Notice for another prob.,

$$\begin{aligned}\mathbb{P}\{\|u\|^2\leqslant 1-\varepsilon\} &= \mathbb{P}\{\exp(-\lambda\|u\|^2)\geqslant \exp(\lambda(-1+\varepsilon))\}\\ &= \mathbb{P}\{\exp(-\lambda\|u\|^2)\exp(\lambda(1-\varepsilon))\geqslant 1\}\\ &\leqslant \exp(\lambda(1-\varepsilon))\mathbb{E}(\exp(-\lambda\|u\|^2))\\ &\to \min_{\lambda>0}\exp(\lambda(1-\varepsilon))\mathbb{E}(\exp(-\lambda\|u\|^2))\\ &= \exp\left(\frac{n(\log(1-\varepsilon)+\varepsilon)}{2}\right)\\ &\overset{(*)_3}{\leqslant} \exp\left(\frac{n(\log(1+\varepsilon)-\varepsilon)}{2}\right)\\ &\leqslant \exp\left(\frac{-n\varepsilon^2}{8}\right)\end{aligned}$$

Hence, $\mathbb{P}\{|\|u\|^2-1|\geqslant\varepsilon\}\leqslant 2\exp\left(\frac{-n\varepsilon^2}{8}\right)$. □

Remarque 2. Here are the proof of the ineq. we used :

$(*)_1\mathbb{E}(\exp(\lambda\|u\|^2))=\left(\frac{n}{n-2\lambda}\right)^{\frac{n}{2}}$: $\chi^2(n)=\Gamma\left(\frac{n}{2},\frac{1}{2}\right)$, sum of $\frac{n}{2}$ $\{X_k\}\overset{i.i.d}{\sim}\text{expo}\left(\frac{1}{2}\right)$, $\mathbb{E}(\exp(\lambda n\|u\|^2))=\prod_{k=1}^{\frac{n}{2}}\frac{\frac{1}{2}}{\frac{1}{2}-\lambda}\Rightarrow\frac{\chi^2(n)}{n}\sim\Gamma\left(\frac{n}{2},\frac{n}{2}\right),, k\in\left[\frac{n}{2}\right]\Rightarrow\mathbb{E}(\exp(\lambda\|u\|^2))=\prod_{k=1}^{\frac{n}{2}}\frac{\frac{n}{2}}{\frac{n}{2}-\lambda}$.

$(*)_2$ $\exp\left(\frac{n(\log(1+\varepsilon)-\varepsilon)}{2}\right)\leqslant\exp\left(\frac{-n\varepsilon^2}{8}\right)$: $\log(1+\varepsilon)-\varepsilon\leqslant-\frac{\varepsilon^2}{4}$.①: $\frac{\log(1+\varepsilon)-\varepsilon}{\varepsilon^2}$ is increasing; ②: $\lim_{\varepsilon\to 1}\frac{\log(1+\varepsilon)-\varepsilon}{\varepsilon^2}\leqslant-\frac{1}{4}$.

$(*)_3\log(1-\varepsilon)+\varepsilon\leqslant\log(1+\varepsilon)-\varepsilon\Rightarrow e^{2\varepsilon}\leqslant\frac{1+\varepsilon}{1-\varepsilon}\Rightarrow 1+2\varepsilon+\frac{4\varepsilon^2}{2}+o(\varepsilon^3)\leqslant\frac{1+\varepsilon}{1-\varepsilon}\Rightarrow o(\varepsilon^3)\leqslant 1+\varepsilon$.

,which comes from Padé approximation.

This lemma tells us the high-dimention vectors sampled from a 'small variance' are almost distributed around the unit ball.

The next lemma tells us the high-dimention vectors are almost orthgonal to each other:

Lemme 3. $u,v\in\mathbb{R}^n, u_i,v_i\overset{i.i.d}{\sim}\mathcal{N}\left(0,\frac{1}{n}\right),\varepsilon\in(0,1)$:

$$\mathbb{P}(|\langle u,v\rangle|\geqslant\varepsilon)\leqslant 4\exp\left(\frac{-n\varepsilon^2}{8}\right)\tag{5}$$

Démonstration. $\mathbb{P}(\langle u,v\rangle\geqslant\varepsilon)=\mathbb{P}(\langle u,v\rangle\geqslant\varepsilon)$.By polarization equality,

$\mathbb{P}(\langle u,v\rangle\geqslant\varepsilon)=\mathbb{P}\left(\frac{\|u+v\|^2-\|u-v\|^2}{4}\geqslant\varepsilon\right)$.Notice if $X,Y\overset{i.i.d}{\sim}\mathcal{N}\left(0,\frac{1}{n}\right)\Rightarrow\frac{X\pm Y}{\sqrt{2}}\sim\mathcal{N}\left(0,\frac{1}{n}\right)\Rightarrow$ $\mathbb{P}\left(\left\|\frac{u+v}{\sqrt{2}}\right\|^2-1\geqslant\varepsilon\right)=\mathbb{P}\left(-\left\|\frac{u-v}{\sqrt{2}}\right\|^2+1\geqslant\varepsilon\right)\leqslant\exp\left(\frac{-n\varepsilon^2}{8}\right)\Rightarrow\mathbb{P}\left(\frac{\|u+v\|^2-\|u-v\|^2}{4}\geqslant\varepsilon\right)\leqslant 2\exp\left(\frac{-n\varepsilon^2}{8}\right)$.Nearly the same way: $\mathbb{P}(\langle u,v\rangle\leqslant\varepsilon)\leqslant 2\exp\left(\frac{-n\varepsilon^2}{8}\right)\Rightarrow\mathbb{P}(|\langle u,v\rangle|\geqslant\varepsilon)\leqslant 4\exp\left(\frac{-n\varepsilon^2}{8}\right)$. □

One application is $U=(v_{\text{ij}})_{i,j\in[n]}, v_{\text{ij}}\sim\mathcal{N}\left(0,\frac{1}{n}\right)^n\Rightarrow U$ is almost *a* orthogonal matrix.

Now we solve the JL lemma, intuition is the almost orthogonal matrix is almost isometric.

Lemme 4. *For N vector* $\{v_1,\ldots,v_N\}\in\mathbb{R}^m$, set $n\geqslant\frac{24\mathrm{logN}}{\varepsilon^2}$, sample the vecter forms $Q=(q_{\mathrm{ij}})_{i\in[n],j\in[m]}$, $q_{\mathrm{ij}}\sim\mathcal{N}\left(0,\frac{1}{n}\right)$, $\varepsilon\in(0,1)$. Then for at least $\frac{N-1}{N}$ prob, $i\neq j$:

$$(1-\varepsilon)\|v_i-v_j\|^2\leqslant\|Q\,v_i-Q\,v_j\|^2\leqslant(1+\varepsilon)\|v_i-v_j\|^2. \tag{6}$$

Démonstration. Easy to show $Q\,u\sim\mathcal{N}\left(0,\frac{1}{n}\right)^n$, for $u\in\mathbb{R}^m$ and $\|u\|_2=1.\Rightarrow\mathbb{P}\left\{\left|\left\|\frac{Q\,(v_i-v_j)}{v_i-v_j}\right\|^2-1\right|\geqslant\varepsilon\right\}\leqslant 2\exp\left(\frac{-n\varepsilon^2}{8}\right)\Rightarrow\mathbb{P}\left\{\exists i,j,s.t.\left|\left\|\frac{Q\,(v_i-v_j)}{v_i-v_j}\right\|^2-1\right|\geqslant\varepsilon\right\}\leqslant 2\binom{N}{2}\exp\left(\frac{-n\varepsilon^2}{8}\right)\Rightarrow$

$$\begin{aligned}\mathbb{P}\left\{\forall i,j,s.t.\left|\left\|\frac{Q\,(v_i-v_j)}{v_i-v_j}\right\|^2-1\right|\leqslant\varepsilon\right\}&=1-\mathbb{P}\left\{\exists i,j,s.t.\left|\left\|\frac{Q\,(v_i-v_j)}{v_i-v_j}\right\|^2-1\right|\geqslant\varepsilon\right\}\\&\geqslant 1-2\binom{N}{2}\exp\left(\frac{-n\varepsilon^2}{8}\right)\\\left(\text{since}\,n\geqslant\frac{24\mathrm{logN}}{\varepsilon^2}\right)&\geqslant 1-\frac{N(N-1)}{N^3}.\\&=1-\frac{1}{N}+\frac{1}{N^2}\\N>0&\geqslant 1-\frac{1}{N}\end{aligned}$$

,which forms the proof.