Set $J(x) := \int_0^T L(t, x, \dot{x}) \mathrm{dt}$. Since $x + \alpha\eta = x$ when $t = 0, T \Rightarrow \eta(0) = \eta(T) = 0$.

$$\begin{aligned} J(x+\alpha\eta) &= \int_0^T L(t, x+\alpha\eta, \dot{x}+\alpha\dot{\eta}) \mathrm{dt} \\ &= \int_0^T L(t, x, \dot{x}) + \left(\int_0^T \frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta} \right) \times \alpha \mathrm{dt} + o(\alpha^2) \\ (\text{integration by part}) &= \int_0^T L(t, x, \dot{x}) \mathrm{dt} + \int_0^T \left(\frac{\partial L}{\partial x}\eta - \frac{d\frac{\partial L}{\partial \dot{x}}}{\mathrm{dt}}\eta \right) \alpha \mathrm{dt} + \frac{\partial L}{\partial \dot{x}}\eta\alpha|_0^T + o(\alpha^2) \\ (\text{Since } \eta(0) = \eta(T) = 0) &= J(x) + \int_0^T \left(\frac{\partial L}{\partial x} - \frac{d\frac{\partial L}{\partial \dot{x}}}{\mathrm{dt}} \right) \eta\alpha \mathrm{dt} + o(\alpha^2). \end{aligned}$$

So

$$DJ(x)[a\eta] = \int_0^T \left(\frac{\partial L}{\partial x} - \frac{d\frac{\partial L}{\partial \dot{x}}}{\mathrm{dt}} \right) \eta\alpha \mathrm{dt} \overset{\mathrm{rep}}{=} \left\langle \frac{\partial L}{\partial x} - \frac{d\frac{\partial L}{\partial \dot{x}}}{\mathrm{dt}}, a\eta \right\rangle. \tag{1}$$

We can choose $\eta < 0$ when $\frac{\partial L}{\partial x} - \frac{d\frac{\partial L}{\partial \dot{x}}}{\mathrm{dt}} < 0$, so the positive property for inner product is still maintained. And for bilinear and symmetric properties are trivial.

Therefore, $\boxed{\delta J = \frac{\partial L}{\partial x} - \frac{d\frac{\partial L}{\partial \dot{x}}}{\mathrm{dt}}.}$

$F = m\ddot{x} = -\frac{\partial U}{\partial x}, K = \frac{1}{2}m\dot{x}^2. L = U - K, \frac{\partial L}{\partial x} = \frac{\partial U}{\partial x}, \frac{d\left(\frac{\partial L}{\partial \dot{x}}\right)}{\mathrm{dt}} = \frac{\mathrm{dm}\dot{x}}{\mathrm{dt}} = m\ddot{x} \Rightarrow \frac{\partial L}{\partial x} = \frac{d\left(\frac{\partial L}{\partial \dot{x}}\right)}{\mathrm{dt}}.$